NOTIZ

Local Instabilities in Acoustic-Gravity Waves with Shear *

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The problem of small amplitude waves propagating horizontally in an isothermal compressible fluid with constant parallel shear flow and within a constant gravitational field is treated. The waves are shown to be locally unstable in the "gravity-wave-like" mode for shear strengths above a critical value.

Conjectured by TAYLOR [1], and proved by MILES [2] and HOWARD [3], the *sufficient* conditions for stability in an isothermal incompresible fluid of variable density with parallel shear flow is that

$$J > \tfrac{1}{4},$$

where J is the local Richardson number. The eigenvalue problem was solved by assuming spatially periodic wave motion subject to fixed boundary conditions. CASE [4] and DYSON [5] solved a similar incompressible fluid problem as an initial value problem. Recently, WARREN [6] has treated the compressible problem in the Miles-Howard manner with a "rigid lid" type boundary condition. He found the *sufficient* condition for stability to be

$$J > \tfrac{1}{4}(1 + M^2),$$

where

$$M = (v_{0\,\max} - v_{0\,\min})/a.$$

$v_{0\,\max}$ and $v_{0\,\min}$ are the maximum und minimum values of the fluid flow between two parallel horizontal rigid planes. a is the sound velocity.

An important point in considering the boundary-value or initial-value problem is that the solution is restricted to certain eigen-modes which belong to a continuous spectrum. Therefore, in this paper we look for instabilities in the *local* dispersion relation for an isothermal *compressible* fluid of constant parallel shear flow and exponentially decreasing density.

The fundamental equations necessary to describe the wave motions are:

a) Euler Equation

$$\varrho\left[\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\nabla)\,\boldsymbol{v}\right] = -\nabla p + \boldsymbol{g}, \tag{1}$$

where $\boldsymbol{v}$ is the velocity vector of a fluid element of the oscillating medium, p is the pressure, ϱ the density, and $\boldsymbol{g}$ is the gravity vector.

b) The Continuity Equation

$$\frac{\partial \varrho}{\partial t} + \nabla\cdot(\varrho\,\boldsymbol{v}) = 0. \tag{2}$$

c) Equation of State (adiabatic approximation)

$$\frac{\mathrm{d}}{\mathrm{d}t}(p\,\varrho^{-\gamma}) = 0, \tag{3}$$

where γ is the ratio of specific heats. Expressing the time derivative in Eq. (3) in terms of quantities referring to points fixed in space, we have

$$\frac{\partial p}{\partial t} - \gamma\,\frac{p}{\varrho}\,\frac{\partial \varrho}{\partial t} + \varrho^{\gamma}(\boldsymbol{v}\cdot\nabla)(p\,\varrho^{-\gamma}) = 0. \tag{4}$$

For the equilibrium conditions, we have

$$\varrho_{\text{eq.}} = \varrho_0\, e^{-z/h}, \tag{5}$$

$$p_{\text{eq.}} = p_0\, e^{-z/h}, \tag{6}$$

and

$$\boldsymbol{v} = v_0(z)\,\boldsymbol{e}_x, \tag{7}$$

where $h = a^2/g\,\gamma$ is the "scale height", and $\boldsymbol{e}_x$ is the unit vector in the x-direction.

We assume that all perturbed quantities are small. We put

$$\varrho(\boldsymbol{r},t) = [\varrho_0 + \varrho'(\boldsymbol{r},t)]\, e^{-z/h}, \tag{8}$$

$$p(\boldsymbol{r},t) = [p_0 + p'(\boldsymbol{r},t)]\, e^{-z/h}, \tag{9}$$

and

$$\boldsymbol{v}(\boldsymbol{r},t) = \boldsymbol{v}_0(z) + \boldsymbol{v}'(\boldsymbol{r},t). \tag{10}$$

By the substitutions of Eqs. (8), (9), and (10) into the fundamental Eqs. (1), (2), and (4), and by making use of the equilibrium conditions (5), (6), and (7), we obtain, after linearization, three linear differential equations with non-constant coefficients. If we assume that the coefficients are exactly constant, which means that we have a velocity profile independent of z, that is, the velocity gradient is zero, then our system under a linear transformation reduces to that of internal gravity waves in a stationary medium, which has already been solved [7]. However, in the next lowest non-trivial approximation, we assume that the vertical wavelength is small compared to the change in velocity profile. This allows us to treat k_z large compared to the change in velocity profile when taking the Fourier transform of the linearized perturbed equations. Performing

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the FOURIER transform, we have for the Euler equation, the continuity equation, and the equation of state, respectively

$$-i\,\omega\,\varrho_0\boldsymbol{v}_c+i(\boldsymbol{k}\cdot\boldsymbol{e}_x)\,\varrho_0\,\boldsymbol{v}_c\,v_0+\varrho_0\,\frac{\mathrm{d}v_0}{\mathrm{d}z}\,(\boldsymbol{v}_c\cdot\boldsymbol{e}_z)\boldsymbol{e}_x - \frac{p_c}{h}\,\boldsymbol{e}_z+i\boldsymbol{k}\,p_c-\varrho_c\,\boldsymbol{g}=0\,, \tag{11}$$

$$\omega\,\varrho_c-\varrho_0(\boldsymbol{k}\cdot\boldsymbol{v}_c)-v_0(\boldsymbol{k}\cdot\boldsymbol{e}_x)\,\varrho_c-i\,\frac{\varrho_0}{h}\,(\boldsymbol{v}_c\cdot\boldsymbol{e}_z)=0\,, \tag{12}$$

$$\omega\,p_c-\omega\,a^2\,\varrho_c+i\,\frac{\gamma-1}{h}\,p_0(\boldsymbol{v}_c\cdot\boldsymbol{e}_z)-v_0\,p_c(\boldsymbol{k}\cdot\boldsymbol{e}_x) + a^2(\boldsymbol{k}\cdot\boldsymbol{e}_x)\,v_0\,\varrho_c=0\,, \tag{13}$$

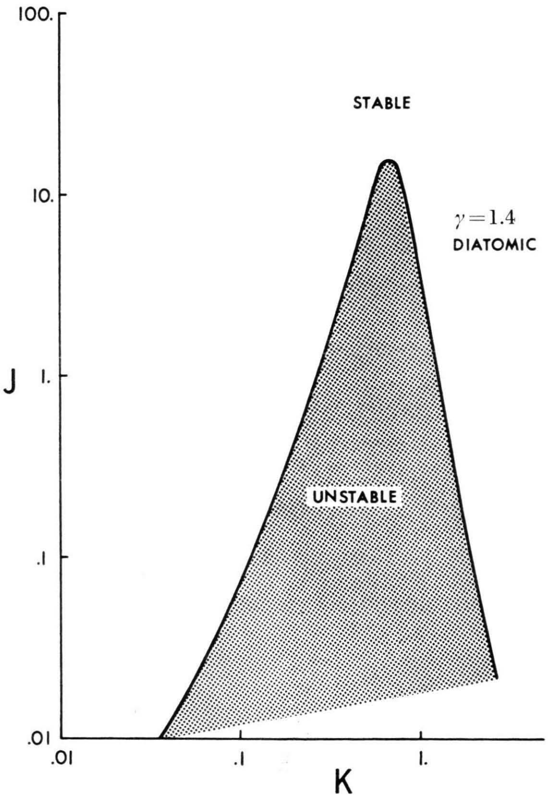

Fig. 1. Richardson number—wave number (shear—wavelength) plot showing the region of instability for diatomic gases ($\gamma=1.4$).

where $\boldsymbol{e}_z$ is the unit vector in the z direction. ϱ_c and p_c are constants with units of density and pressure, respectively. $\boldsymbol{v}_c$ is a constant vector in the direction of $\boldsymbol{v}'$. $\boldsymbol{k}$ is the propagating wave number vector, and ω is the wave frequency.

Equations (11), (12), and (13) comprise a system of five scalar equations and five unknowns; v_{cx}, v_{cy}, v_{cz}, ϱ_c and p_c. The condition for a nontrivial solution is obtained by setting the determinant of the coefficients of the unknowns equal to zero. This yields the local dispersion relation.

We consider a linear transformation to the intrinsic frequency moving with the fluid

$$\bar{\omega}=\omega-v_0(\boldsymbol{k}\cdot\boldsymbol{e}_x)\,, \tag{14}$$

and a linear transformation in k-space to eliminate the amplitude growth due to the exponential decay of density with height.

$$k_x=K_x\,,\qquad k_y=K_y\,,\qquad k_z=K_z-i\,\frac{g\,\gamma}{2\,a^2}\,, \tag{15}$$

and furthermore

$$K^2=K_x^2+K_y^2+K_z^2.$$

In non-dimensional form, the dispersion relation is

$$W^4+W^2[-\varkappa^2-(\tfrac{1}{2}\,\gamma)^2] + W\{[i(\boldsymbol{\varkappa}\cdot\boldsymbol{e}_z)-\tfrac{1}{2}\,\gamma+1]\;S(\boldsymbol{\varkappa}\cdot\boldsymbol{e}_x)\} + (\gamma-1)\,[\varkappa^2-(\boldsymbol{\varkappa}\cdot\boldsymbol{e}_z)^2]=0\,, \tag{16}$$

where

$$W=\frac{a}{g}\,\bar{\omega}\,,\quad \varkappa=\frac{a^2}{g}\,K\,,\quad\text{and}\quad S=\frac{a}{g}\,\frac{\mathrm{d}v_0}{\mathrm{d}z}\,.$$

Let us assume propagation along the horizontal (x direction), then Eq. (16) has four roots, one of which exhibits unstable growth for a range of wavelengths dependent upon the shear strength. This unstable mode is a „gravity-like" wave propagating with the wind for positive shear. The onset of instability, which is determined by the appearance of a positive imaginary part to this root, occurs at the Richardson number

$$J_c=\frac{\gamma(\tfrac{1}{2}\,\gamma-1)^2}{[(\gamma-1)-(\tfrac{1}{2}\,\gamma)^2]^2}\,. \tag{17}$$

For diatomic gases (see Fig. 1), this critical Richardson number is

$$J_c=15.55\,.$$

In this analysis, we have used the general definition of the Richardson number, i. e. the ratio of the buoyancy force to the inertial force

$$J=\frac{-g}{\varrho}\,\frac{\mathrm{d}\varrho/\mathrm{d}z}{(\mathrm{d}v_0/\mathrm{d}z)^2}\,.$$

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